

CANTOR TYPE FUNCTIONS IN NON-INTEGER BASES

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ABSTRACT. Cantor's ternary function is generalized to arbitrary base-change functions in non-integer bases. Some of them share the curious properties of Cantor's function, while others behave quite differently.

1. INTRODUCTION

Cantor's celebrated ternary function has been constructed by combining expansions of real numbers in two different bases $p = 3$ and $q = 2$. This function has the surprising property to be a non-constant, continuous and non-decreasing function, having a zero derivative almost everywhere. We refer to [19] for a survey of this and various other interesting features of Cantor's function.

The purpose of this paper is to put Cantor's construction into a general framework by considering arbitrary bases $p, q > 1$. It turns out that for some values of p and q these functions have similar properties, while for other values they exhibit a quite different behavior.

Given a real *base* $p > 1$, by an *expansion* of a real number x in base p we mean a sequence $c = (c_i) \in \{0, 1\}^\infty$ satisfying the equality

$$(1.1) \quad \pi_p(c) := \sum_{i=1}^{\infty} \frac{c_i}{p^i} = x.$$

We denote by J_p the set of numbers x having at least one expansion. It is clear that $J_p \subseteq [0, \frac{1}{p-1}]$.

For $p = 2$ the definition reduces to the familiar binary expansions of the numbers $x \in [0, 1]$. If $x \in (0, 1)$ is a binary rational number, then it has two expansions: one ending with 0^∞ , and another one ending with 1^∞ .¹ The remaining numbers in $[0, 1]$ have a unique expansion.

The case $1 < p < 2$ was first investigated by Rényi [31]. He has proved among others the equality $J_p = [0, \frac{1}{p-1}]$ for each $p \in (1, 2]$ by applying a greedy algorithm for each $x \in [0, \frac{1}{p-1}]$ as follows. If b_i has already been

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¹We apply the notations of symbolic dynamics, i.e. d^∞ denotes the constant sequence d, d, \dots , and 10^∞ denotes the sequence $1, 0, 0, \dots$.

defined for all $i < n$ (no assumption if $n = 1$), then we set $b_n = 1$ if

$$(1.2) \quad \left(\sum_{i=1}^{n-1} \frac{b_i}{p^i} \right) + \frac{1}{p^n} \leq x,$$

and $b_n = 0$ otherwise. Then $b_p(x) := (b_i)$ is an expansion of x in base p .² By construction this is the lexicographically largest expansion of x in base p , called the *greedy* or β -*expansion* of x in base p .

Today there is a huge literature devoted to non-integer expansions. For example, probabilistic and ergodic aspects are investigated in [3], [5], [31], [32], [34], combinatorial properties in [1], [4], [18], [25], [26], [30], unique expansions in [6], [7], [8], [9], [10], [12], [14], [15], [16], [21], [22], [23], [24], [27], and control-theoretical applications are given in [2], [28], [29].

Many other references are given in the surveys [11], [20], [33].

The situation for $1 < p < 2$ is quite different from that of $p = 2$. For example, while in base $p = 2$ each $x \in [0, 1]$ has one or two expansions, in the bases $1 < p < 2$ almost every $x \in J_p$ has 2^{\aleph_0} expansions by a theorem of Sidorov [32].

Another difference is that the functions $x \mapsto b_p(x)$ of J_p into $\{0, 1\}^\infty$ are not monotone if $1 < p < 2$, and their behavior depends critically on the value of p . In order to understand their mutual behavior we introduce for $p \in (1, 2]$ and $q > 1$ the *base-change functions*

$$b_{p,q} := \pi_q \circ b_p : J_p \rightarrow J_q.$$

Explicitly, we have

$$(1.3) \quad b_{p,q}(x) := \sum_{i=1}^{\infty} \frac{b_i}{q^i} \quad \text{with} \quad (b_i) = b_p(x).$$

We exclude henceforth the trivial case $p = q$ when we get the identity function on J_p .

Now we may state our first result:

Theorem 1.1. *Let $p \in (1, 2]$ and $q > 1$.*

- (i) *The function $b_{p,q}$ is right continuous. It is left continuous in $x \in J_p$ if and only if $b_p(x)$ contains infinitely many 1 digits. Its discontinuities form a countable dense set.*
- (ii) *If $q < p$, then $b_{p,q}$ is nowhere monotone, not left differentiable anywhere, and not right differentiable in x if the greedy expansion $b_p(x)$ contains at most finitely many non-zero digits.*
- (iii) *If $q > p$, then $b_{p,q}$ is increasing, and hence differentiable almost everywhere.*

Daróczy and Kátai [6], [7] have introduced a slight variant of the β -expansions where the inequality “ \leq ” in (1.2) is changed to the strict inequality “ $<$ ”. More precisely, for $x = 0$ we set $a_p(x) := 0^\infty$. For $x \in (0, \frac{1}{p-1}]$

²Sometimes we write $b_i(x, p)$ instead of b_i to show the dependence on x and p .

we define a sequence $a_p(x) := (a_i)$ as follows. If a_i has already been defined for all $i < n$ (no assumption if $n = 1$), then we set $a_n = 1$ if

$$\left(\sum_{i=1}^{n-1} \frac{a_i}{p^i} \right) + \frac{1}{p^n} < x,$$

and $a_n := 0$ otherwise. Then $a_p(x)$ is again an expansion of x in base p ; it is called its *quasi-greedy expansion*.

In the present context it is customary to call a sequence *finite* if it ends with 10^∞ , and *infinite* otherwise: hence the infinite sequences are 0^∞ and the sequences containing infinitely many 1 digits. Using this terminology, $a_p(x)$ is the lexicographically largest *infinite* expansion of x in base p .

Parry [30] gave a lexicographic characterization of greedy expansions by using the quasi-greedy expansion $a_p(1)$ of $x = 1$, and a similar characterization of quasi-greedy expansions was given in [1]:

Theorem. *Let $p \in (1, 2]$ and $(c_i) \in \{0, 1\}^\infty$.*

- (i) *We have $(c_i) = b_p(x)$ for some $x \in [0, \frac{1}{p-1}]$ if and only if*

$$c_{n+1}c_{n+2} \cdots < a_p(1) \quad \text{whenever} \quad c_n = 0.$$

- (ii) *We have $(c_i) = a_p(1)$ for some $x \in [0, \frac{1}{p-1}]$ if and only if*

$$c_{n+1}c_{n+2} \cdots \leq a_p(1) \quad \text{whenever} \quad c_n = 0.$$

Remark 1.2. *We have $a_p(x) \neq b_p(x)$ if and only if $b_p(x)$ is finite. Hence the two expansions differ only for countably many values of x . These values are dense in J_p . Indeed, for any fixed $x \in J_p$ the truncated finite sequences*

$$b_1(x, p) \cdots b_n(x, p)0^\infty, \quad n = 1, 2, \dots$$

are greedy expansions by Parry's theorem, say

$$b(x_n, p) = b_1(x, p) \cdots b_n(x, p)0^\infty, \quad n = 1, 2, \dots,$$

and then $x_n \rightarrow x$.

Observe that the discontinuity points of $b_{p,q}$ are exactly the points x where $a_p(x) \neq b_p(x)$.

There is a variant of Theorem 1.1 for quasi-greedy expansions. We introduce for $p \in (1, 2]$ and $q > 1$ the *quasi-greedy base-change functions*

$$a_{p,q} := \pi_q \circ a_p : J_p \rightarrow J_q.$$

Explicitly, we have

$$a_{p,q}(x) := \sum_{i=1}^{\infty} \frac{a_i}{q^i} \quad \text{with} \quad (a_i) = a_p(x).$$

We again exclude the trivial case $p = q$.

Theorem 1.3. *Let $p \in (1, 2]$ and $q > 1$.*

- (i) *The function $a_{p,q}$ is left continuous. It is right continuous in $x \in [0, \frac{1}{p-1})$ if and only if $a_p(x) = b_p(x)$.*

- (ii) If $q < p$, then $a_{p,q}$ is nowhere monotone, not left differentiable anywhere, and not right differentiable in x if the greedy expansion $b_p(x)$ contains at most finitely many non-zero digits.
- (iii) If $q > p$, then $a_{p,q}$ is increasing, and hence differentiable almost everywhere.

The statements (ii), (iii) of Theorems 1.1 and 1.3 are similar, the proofs of the crucial properties (ii) are quite different: see the proof of Proposition 4.2 below.

Now we turn to the case $p > 2$. In this case J_p is a proper subset of $[0, \frac{1}{p-1}]$, and each $x \in J_p$ has a unique expansion. Indeed, if $(c_i), (d_i)$ are two different sequences with $c_1 \cdots c_{n-1} = d_1 \cdots d_{n-1}$ and $c_n > d_n$ for some $n \geq 1$, then

$$\sum_{i=1}^{\infty} \left(\frac{c_i}{p^i} - \frac{d_i}{p^i} \right) \geq \frac{1}{p^n} - \sum_{i=n+1}^{\infty} \frac{1}{p^i} = \frac{p-2}{p^n(p-1)} > 0.$$

We will clarify its topological nature of J_p :

Theorem 1.4. *Let $p > 2$.*

- (i) J_p is a Cantor set of Hausdorff dimension $\frac{1}{\log p}$, and hence a null set.
- (ii) $x \in J_p$ is a right accumulation point of J_p if and only if its unique expansion has infinitely many zero digits.
- (iii) $x \in J_p$ is a left accumulation point of J_p if and only if its unique expansion has infinitely many one digits.

Denoting by $b_p(x) = (b_i)$ the unique expansion of x in base $p > 2$, we may extend the definition $b_{p,q} := \pi_q \circ b_p : J_p \rightarrow J_q$ for all $p, q > 1$; explicitly,

$$b_{p,q}(x) := \sum_{i=1}^{\infty} \frac{b_i}{q^i} \quad \text{with} \quad (b_i) = b_p(x).$$

However, the behavior of the functions $b_{p,q}$ is quite different for $p > 2$:

Theorem 1.5. *Let $p > 2$ and $q > 1$.*

- (i) The function $b_{p,q}$ is Hölder continuous with the exact exponent $\frac{\log q}{\log p}$.
- (ii) If $q < p$, then $b_{p,q}$ is nowhere monotone.
- (iii) If $q = 2$, then $b_{p,q}$ is non-decreasing.
- (iv) If $q > 2$, then $b_{p,q}$ is increasing.

Note that for $q > p > 2$ the Hölder exponent in (i) is greater than one. This does not contradict the classical result that only constant functions have Hölder exponents greater than one, because the domain J_p of $b_{p,q}$ is not an interval if $p > 2$.

If $p > 2$, then J_p is a closed set in $[0, \frac{1}{p-1}]$, containing the endpoints of this interval. Therefore we may extend the function $b_{p,q} : J_p \rightarrow J_q$ to a continuous function defined on $[0, \frac{1}{p-1}]$. Adapting the construction of

Cantor's ternary function (it corresponds to the case $p = 3$ and $q = 2$) we introduce the continuous and surjective function

$$B_{p,q} : \left[0, \frac{1}{p-1}\right] \rightarrow \left[0, \frac{1}{q-1}\right]$$

that coincides with $b_{p,q}$ on J_p , and is affine on each connected component of $[0, \frac{1}{p-1}] \setminus J_p$.

Theorem 1.6. *Let $p > 2$ and $q > 1$.*

- (i) *The function $B_{p,q}$ is differentiable almost everywhere, and is Hölder continuous with the exact exponent $\min \left\{ 1, \frac{\log q}{\log p} \right\}$.*
- (ii) *If $q < 2$, then $B_{p,q}$ has no bounded variation.*
- (iii) *If $q = 2$, then $B_{p,q}$ is non-decreasing, but not absolutely continuous. Its arc length is equal to $p/(p-1)$.*
- (iv) *If $q > 2$, then $B_{p,q}$ is increasing, absolutely continuous, and $B'_{p,q} > 0$ a.e.*

The theorem shows that the above mentioned properties of Cantor's function remain valid for $q = 2$ and all $p > 2$, but not for $q \neq 2$.

The proof of the theorem (see equation (4.3) below) will also yield an explicit formula for the derivative. If we denote by $I_{m,k}$, $k = 1, \dots, 2^m$ the removed intervals at the m th step of the construction of $B_{p,q}$ ($m = 0, 1, \dots$), then

$$B'_{p,q}(x) = \left(\frac{p}{q}\right)^{m+1} \frac{(q-2)(p-1)}{(p-2)(q-1)}, \quad x \in I_{m,k}$$

for all m and k .

The remainder of the paper is devoted to the proof of the above statements. They may be easily adapted to the case of more general digits sets $\{0, 1, \dots, M\}$ with a given positive integer, by distinguishing the intervals $(1, M+1)$ and $(M+1, \infty)$ instead of $(1, 2)$ and $(2, \infty)$.

2. STUDY OF THE MAP π_p

The set $\{0, 1\}^\infty$ of sequences $c = (c_i)$ is compact for the Tikhonov product topology, induced by the following metric: $\rho(c, d) = 0$ if $c = d$, and $\rho(c, d) = 2^{-n}$ if $c_1 \cdots c_{n-1} = d_1 \cdots d_{n-1}$ and $c_n \neq d_n$. The corresponding convergence is the *coordinate-wise convergence*.

The set $\{0, 1\}^\infty$ also has a natural *lexicographic order* and a corresponding *order topology*. The two topologies coincide:

Proposition 2.1. *The product topology and the order topology coincide on $\{0, 1\}^\infty$.*

Proof. Each open ball is a finite intersection of open intervals, and hence open in the order topology:

$$B_{2^{-n}}(c) = \cap_{j=1}^n A_j(c)$$

with

$$A_j(c) := \begin{cases} \{d \in \{0,1\}^\infty : d_1 \cdots d_{j-1} > 01^\infty\} & \text{if } c_j = 1, \\ \{d \in \{0,1\}^\infty : d_1 \cdots d_{j-1} < 10^\infty\} & \text{if } c_j = 0, \end{cases} \quad j = 1, \dots, n.$$

Conversely, each interval of the form

$$A(c) := \{d \in \{0,1\}^\infty : d > c\} \quad \text{or} \quad \tilde{A}(c) := \{d \in \{0,1\}^\infty : d < c\}$$

(they form a subbase for the order topology) is open in the metric topology. Indeed, for any fixed $d \in A(c)$ there exists an integer $n \geq 1$ such that $d_1 \cdots d_n > c_1 \cdots c_n$, and then $B_{2^{-n-1}}(d) \subset A(c)$ because

$$f \in B_{2^{-n-1}}(d) \implies f_1 \cdots f_n = d_1 \cdots d_n > c_1 \cdots c_n.$$

The proof for $\tilde{A}(c)$ is analogous. \square

Henceforth we consider this metric and topology on $\{0,1\}^\infty$.

Proposition 2.2. *The Hausdorff dimension of $\{0,1\}^\infty$ is equal to 1.*

Proof. The whole space is the compact invariant set of the iterated function system consisting of the two similarities of ratio $1/2$, defined by the formulas

$$f_i(c_1 c_2 \cdots) := (i c_1 c_2 \cdots), \quad i = 0, 1.$$

Since their images are disjoint, Applying [13, Theorem 6.4.3] (see also [17]) we obtain that the Hausdorff dimension is the solution of the equation

$$2^{-d} + 2^{-d} = 1,$$

i.e., $d = 1$. \square

Next we investigate the function $\pi_p : \{0,1\}^\infty \rightarrow J_p$ introduced in (1.1). In the following proposition and in the sequel we use base two logarithm.

Proposition 2.3. *Let $q > 1$.*

- (i) π_q is continuous, and even Hölder continuous with the exact exponent $\alpha = \log p$; more precisely,

$$(2.1) \quad |\pi_q(c) - \pi_q(d)| \leq \frac{q}{q-1} \rho(c, d)^{\log q}$$

for all sequences c et d . Hence its range J_q is a non-empty compact set.

- (ii) If $1 < q < 2$, then $J_q = \left[0, \frac{1}{q-1}\right]$, and π_q is nowhere monotone.
 (iii) If $q = 2$, then $J_q = [0, 1]$, and π_q is non-decreasing.
 (iv) If $q > 2$, then π_q is an increasing homeomorphism of $\{0,1\}^\infty$ onto J_q .
 Moreover, we have a converse inequality to (2.1) :

$$(2.2) \quad \frac{q-2}{q-1} \rho(c, d)^{\log q} \leq |\pi_q(c) - \pi_q(d)|$$

for all sequences c et d .

Hence J_q has Hausdorff dimension $\frac{1}{\log q} < 1$ and therefore it is a null set.

Proof. (i) We prove the Hölder continuity of π_q . If $\rho(c, d) = 2^{-n}$, then

$$c_1 \cdots c_{n-1} = d_1 \cdots d_{n-1}$$

and therefore

$$|\pi_q(c) - \pi_q(d)| \leq \sum_{i=n}^{\infty} \frac{1}{q^i} = \frac{1}{q^{n-1}(q-1)} = \frac{q}{q-1} \rho(c, d)^{\log q}.$$

The exponent $\alpha = \log q$ cannot be improved because for $c = 1^\infty$ and $d = 0^\infty$ we have equality.

Since $\{0, 1\}^\infty$ is compact and π_q is continuous, the range J_q of π_q is also compact.

(ii) We already know that $J_q = \left[0, \frac{1}{q-1}\right]$.

Given an arbitrary non-degenerate subinterval I of $\left[0, \frac{1}{q-1}\right]$, there exist a large integer n and a block $a_1 \cdots a_{n-1} \in \{0, 1\}^{n-1}$ such that $\pi_q(c) \in I$ for all sequences c starting with $a_1 \cdots a_{n-1}$. Consider the sequences

$$b := a_1 \cdots a_{n-1} 0^\infty, \quad c := a_1 \cdots a_{n-1} 01^\infty \quad \text{and} \quad d := a_1 \cdots a_{n-1} 10^\infty.$$

We have

$$b < c < d \quad \text{and} \quad \pi_q(b), \pi_q(c), \pi_q(d) \in I.$$

Furthermore, we have obviously $\pi_q(b) < \pi_q(c)$, and also $\pi_q(c) > \pi_q(d)$ because

$$\pi_q(d) - \pi_q(c) = \frac{1}{q^n} - \sum_{i=n+1}^{\infty} \frac{1}{q^i} = \frac{(q-2)}{q^n(q-1)} < 0.$$

Hence π_q is not monotone in I .

(iii) The property $J_2 = [0, 1]$ is well known.

If two sequences $c < d$ first differ at their n th digits, then

$$\pi_2(d) - \pi_2(c) \geq \frac{1}{2^n} - \sum_{i=n+1}^{\infty} \frac{1}{2^i} = 0.$$

Hence π_2 is non-decreasing.

It is not (strictly) increasing because $\pi_2(10^\infty) = \pi_2(01^\infty)$.

(iv) If $p > 2$, and two sequences $c < d$ first differ at their n th digits, then

$$\pi_q(d) - \pi_q(c) \geq \frac{1}{q^n} - \sum_{i=n+1}^{\infty} \frac{1}{q^i} = \frac{(q-2)}{q^n(q-1)} > 0.$$

This proves the increasingness of π_q and the inequality (2.2).

It follows from (2.1) and (2.2) that π_q is a homeomorphism.

Finally, since $\{0, 1\}^\infty$ has Hausdorff dimension 1, using (2.1) and (2.2) we infer from the definition of the Hausdorff dimension that J_q has Hausdorff dimension $1/\log q < 1$. \square

Next we investigate the greedy and quasi-greedy maps $b_p : J_p \rightarrow \{0, 1\}^\infty$ and $a_p : J_p \rightarrow \{0, 1\}^\infty$, defined in the introduction. We have

$$\pi_p(b_p(x)) = \pi_p(a_p(x)) = x \quad \text{for all } x \in J_p$$

by definition, i.e., π_p is a left inverse of both b_p and a_p .

Proposition 2.4. *We have*

$$1 < p < q \leq 2 \implies R(b_p) \subsetneq R(b_q) \quad \text{and} \quad R(a_p) \subsetneq R(a_q),$$

and

$$R(b_p) = \{0, 1\}^\infty \quad \text{for all } p > 2.$$

Proof. The inclusions \subseteq and for $p > 2$ the equalities follow from the definitions of the greedy and quasi-greedy expansions:

$$(c_i) \in R(b_p) \iff \sum_{i=1}^{\infty} \frac{c_{n+i}}{p^i} < 1 \quad \text{whenever } c_n = 0,$$

and

$$(c_i) \in R(a_p) \iff \sum_{i=1}^{\infty} \frac{c_{n+i}}{p^i} \leq 1 \quad \text{whenever } c_n = 0,$$

because both inequalities remain valid if we change p to a larger q .

It remains to show that for $1 < p < q \leq 2$ the sets $R(b_q) \setminus R(b_p)$ and $R(a_q) \setminus R(a_p)$ are non-empty. It follows from [10, Theorem 2.7] that $R(b_q) \setminus R(b_p)$ has in fact 2^{\aleph_0} elements. Since the sets $R(b_q) \setminus R(a_q)$ and $R(a_p) \setminus R(b_p)$ are countable by Remark 1.2, $R(a_q) \setminus R(a_p)$ has also 2^{\aleph_0} elements. \square

We recall that $J_p = \left[0, \frac{1}{p-1}\right]$ is an interval if $p \in (1, 2]$. Now we clarify the topological picture of J_p for $p > 2$. We recall that each $x \in J_p$ has a unique expansion.

Proposition 2.5. *Let $p > 2$ and $x \in J_p$.*

- (i) *$x \in J_p$ is a right accumulation point of J_p if and only if its unique expansion has infinitely many zero digits.*
- (ii) *$x \in J_p$ is a left accumulation point of J_p if and only if its unique expansion has infinitely many one digits.*
- (iii) *J_p is a Cantor set, i.e., a non-empty compact set having neither interior, nor isolated points.*

Proof. (i) If the unique expansion (c_i) of x has infinitely many zero digits $c_n = 0$, then the formula

$$x_n := \left(\sum_{i=0}^{n-1} \frac{c_i}{p^i} \right) + \frac{1}{p^n},$$

where n runs over the integers for which $c_n = 0$, defines a sequence (x_n) converging to x and satisfying $x_n > x$ for all n . Hence x is a right accumulation point.

Otherwise either $x = \max J_p$ or the unique expansion (c_i) of x has a last zero digit $c_m = 0$. In the first case x is obviously a right isolated point. In the second case the unique expansion of each $y \in J_p$, $y > x$ starts with some block $d_1 \cdots d_m > c_1 \cdots c_m$, so that

$$y - x \geq \frac{1}{p^m} - \sum_{i=m+1}^{\infty} \frac{1}{p^i} = \frac{p-2}{p^m(p-1)} > 0.$$

Since the right side does not depend on the particular choice of y , we conclude that x is a right isolated point again.

(ii) The statements (i) and (ii) are equivalent because J_p is symmetric with respect to $\frac{1}{2(p-1)}$, and the expansion of any $x \in J_p$ is the *reflection* $(1 - c_i)$ of the expansion (c_i) of $\frac{1}{p-1} - x$.

(iii) We already know that J_p is a non-empty compact set. We also know that it is a null set, hence J_p has no interior points. Finally, since every sequence has infinitely many equal digits, all points of J_p are accumulation points by (i) and (ii). \square

We end this section by investigating the continuity of the maps b_p and a_p .

Proposition 2.6. *Let $p > 1$ and $x \in J_p$.*

- (i) *If $p \in (1, 2]$, then b_p is right continuous and a_p is left continuous. Furthermore, they are continuous in x if and only if $b_p(x) = a_p(x)$.*
- (ii) *If $p > 2$, then b_p is a homeomorphism between J_p and $\{0, 1\}^\infty$. Moreover,*

$$c_1 |x - y|^{1/\log p} \leq \rho(b_p(x), b_p(y)) \leq c_2 |x - y|^{1/\log p}$$

for all $x, y \in J_p$ with

$$c_1 := \left(\frac{p-1}{p}\right)^{1/\log p} \quad \text{and} \quad c_2 := \left(\frac{p-1}{p-2}\right)^{1/\log p}.$$

Proof. (i) Let $p \in (1, 2]$, and consider a sequence (x_k) converging to x in J_p .

If $x_k > x$ for all k , then $b_p(x_k) \rightarrow b_p(x)$ coordinate-wise: this follows from the definition of the greedy algorithm (or see [10, Lemma 2.5]). This shows that b_p is right continuous.

Furthermore, since

$$b_p(x_k) \geq a_p(x_k) > b_p(x) \geq a_p(x),$$

we have also $a_p(x_k) \rightarrow b_p(x)$ coordinate-wise. Therefore a_p is right continuous in x if and only if $b_p(x) = a_p(x)$.

If $x_k < x$ for all k , then $a_p(x_k) \rightarrow a_p(x)$ coordinate-wise: this follows from the definition of the quasi-greedy algorithm (or see [10, Lemma 2.3]). This shows that a_p is left continuous.

Furthermore, since

$$a_p(x_k) \leq b_p(x_k) < a_p(x) \leq b_p(x),$$

it follows that $b_p(x_k) \rightarrow a_p(x)$ coordinate-wise. Therefore b_p is left continuous in x if and only if $b_p(x) = a_p(x)$.

(ii) This follows from Proposition 2.3 (i) and (iv) because b_p is the inverse of the homeomorphism π_p . \square

Remarks 2.7.

- (i) *We may also give a direct proof of the discontinuity of b_p and a_p at x when $b_p(x) = (b_i)$ has a last non-zero digit $b_m = 1$. We show that $\rho(b_p(y), b_p(x)) \geq 2^{-m} > 0$ for all $y < x$, and $\rho(b_p(y), a_p(x)) \geq 2^{-m} > 0$ for all $y > x$.*

Indeed, for $y < x$ we have $b_p(y) < b_1 \cdots b_m 0^\infty$ and therefore

$$\rho(b_p(y), b_p(x)) \geq 2^{-m} > 0.$$

Similarly, for $y > x$ we have $a_p(y) > b_1 \cdots b_m 0^\infty$. Since $a_p(x) = (b_1 \cdots b_{m-1} 0)^\infty$, this implies that $\rho(a_p(y), a_p(x)) \geq 2^{-m} > 0$.

- (ii) *By Remark 1.2 the discontinuities of b_p and a_p form a countable dense set in J_p .*
- (iii) *If we endow $R(b_p)$ and $R(a_p)$ with the topology associated with the lexicographic order on these sets, then b_p and a_p remain homeomorphisms. This shows that this topology is different from the topology associated with the restriction of the metric ρ : the latter one is finer.*

Indeed, the second part of the proof of Proposition 2.1 remains valid in each subset of $\{0, 1\}^\infty$, but the first may fail. Consider for example the open balls

$$B_{1/2}(0^\infty) = \{c \in \{0, 1\}^\infty : c < 10^\infty\}$$

and

$$B_{1/2}(1^\infty) = \{c \in \{0, 1\}^\infty : c > 01^\infty\}$$

in $\{0, 1\}^\infty$.

Then $B_{1/2}(0^\infty) \cap R(a_p)$ is not open in the order topology of $R(a_p)$ because it has a maximal element 01^∞ , and each order-neighborhood of 01^∞ contains larger sequences in $R(a_p)$.

Similarly, $B_{1/2}(1^\infty) \cap R(b_p)$ is not open in the order topology of $R(b_p)$ because it has a minimal element 10^∞ , and each order-neighborhood of 10^∞ contains smaller sequences in $R(b_p)$.

3. MONOTONICITY AND CONTINUITY OF BASE-CHANGE FUNCTIONS

In this section we investigate the *base-change functions*

$$b_{p,q} := \pi_q \circ b_p : J_p \rightarrow J_q \quad \text{for } p, q > 1,$$

and

$$a_{p,q} := \pi_q \circ a_p : J_p \rightarrow J_q \quad \text{for } p \in (1, 2] \quad \text{and } q > 1.$$

We exclude the trivial case $p = q$ where they are the identity maps of J_p .

First we study $b_{p,q}$.

Proposition 3.1. *Let $p, q > 1$ with $p \neq q$.*

- (i) *If $q > \min\{p, 2\}$, then $b_{p,q}$ is increasing.*
- (ii) *If $q = 2 < p$, then $b_{p,q}$ is non-decreasing.*
- (iii) *In the remaining case $q < \min\{p, 2\}$ the map $b_{p,q}$ is nowhere monotone.*

Proof. (i) If $q > 2$, then both b_p and π_q are increasing, hence $b_{p,q} = \pi_q \circ b_p$ is also increasing.

If $q > p$, then both b_p and the restriction of π_q to $R(b_p) \subseteq R(b_q)$ are increasing, hence $b_{p,q} = \pi_q \circ b_p$ is also increasing.

(ii) If $q = 2 < p$, then b_p is increasing and π_q is non-decreasing, so that $b_{p,q} = \pi_q \circ b_p$ is non-decreasing. It is not increasing, however, because for example $\pi_2(10^\infty) = \pi_2(01^\infty)$, whence $b_{p,2}$ takes the same value at the points $b_p^{-1}(01^\infty) < b_p^{-1}(10^\infty)$.

(iii) Assume henceforth that $q < \min\{p, 2\}$, and fix an arbitrary open interval $I \subseteq [0, \frac{1}{p-1}]$ such that $I \cap J_p \neq \emptyset$.

Fix a point $x \in I \cap J_p$ whose greedy expansion $b_p(x) = (b_i)$ is infinite, and choose a large integer n such that $b_n = 1$, and $\pi_p(c) \in I$ for all p -greedy sequences c starting with $b_1 \cdots b_{n-1}$.

Furthermore, fix a number $r \in (q, \min\{p, 2\})$ and consider the quasi-greedy expansion $(\alpha_i) := a_r(1)$. (It is well defined because $r \in (1, 2)$.)

Since $r < p$,

$$b_1 \cdots b_{n-1} 0^\infty < b_1 \cdots b_{n-1} 0 \alpha_1 \alpha_2 \cdots < b_1 \cdots b_{n-1} 10^\infty$$

are greedy expansions in base p of suitable numbers $x < y < z$ (we use here the increasingness of the map $x \mapsto b_p(x)$). Since $x, y, z \in I$ by the choice of n , the proof will be completed by showing that $\pi_q(y) > \pi_q(x)$ and $\pi_q(y) > \pi_q(z)$.

The first relation is obvious:

$$\pi_q(y) - \pi_q(x) = \sum_{i=1}^{\infty} \frac{\alpha_i}{q^{n+i}} > 0.$$

The second relation follows from our assumption $q < r$:

$$q^n (\pi_q(y) - \pi_q(z)) = \left(\sum_{i=1}^{\infty} \frac{\alpha_i}{q^i} \right) - 1 > \left(\sum_{i=1}^{\infty} \frac{\alpha_i}{r^i} \right) - 1 = 0. \quad \square$$

Corollary 3.2. *Let $p, q > 1$ with $p \neq q$.*

- (i) *$b_{p,q}$ is injective if and only if $q > \min\{p, 2\}$.*
- (ii) *$b_{p,q}$ is onto if and only if $p > \min\{q, 2\}$.*
- (iii) *$b_{p,q}$ is bijective if and only if $\min\{p, q\} > 2$.*

Proof. Part (i) follows from the preceding proposition.

Since π_q is a bijection between $R(b_q)$ and J_q , $b_{p,q} = \pi_q \circ b_p$ is onto if and only if $R(b_q) \subseteq R(b_p)$. In view of Proposition 2.4 this is equivalent to the condition $p > \min\{q, 2\}$. This proves (ii).

Parts (i) and (ii) imply (iii). \square

Next we investigate the continuity:

Proposition 3.3. *Let $p, q > 1$ with $p \neq q$.*

- (i) *If $p, q > 2$, then $b_{p,q}$ is a homeomorphism.*
- (ii) *If $p > 2$, then $b_{p,q}$ is Hölder continuous with the exact exponent $\frac{\log q}{\log p}$.*
- (iii) *If $p \in (1, 2]$, then $b_{p,q}$ is right continuous. It is left continuous in $x \in (0, \frac{1}{p-1}]$ if and only if $b_p(x)$ has infinitely many 1 digits.*

Remark 3.4. *For $q > p > 2$ the Hölder exponent is greater than one. This is possible because b_p is not defined on an interval.*

Proof of Proposition 3.3. Propositions 2.3 and 2.6 imply most statements by the continuity of composite functions. It remains to prove that if $p \in (1, 2]$ and $(b_i) := b_p(x)$ has a last non-zero digit $b_n = 1$, then $b_{p,q}$ is not left continuous in x .

Setting $(\alpha_i) := a_p(1)$ for brevity we have

$$b_p(x) = b_1 \cdots b_{n-1} b_n 0^\infty \quad \text{and} \quad a_p(x) = b_1 \cdots b_{n-1} (b_n - 1) \alpha_1 \alpha_2 \cdots.$$

Furthermore, if $y < x$ and $y \rightarrow x$, then $b_p(y) \rightarrow a_p(x)$ (see the proof of Proposition 2.6 (i)), and therefore

$$\begin{aligned} b_{p,q}(x) - \lim_{y \nearrow x} b_{p,q}(y) &= \frac{1}{q^n} \left(1 - \sum_{i=1}^{\infty} \frac{\alpha_i}{q^i} \right) \\ &\neq \frac{1}{q^n} \left(1 - \sum_{i=1}^{\infty} \frac{\alpha_i}{p^i} \right) \\ &= 0. \end{aligned} \quad \square$$

Remark 3.5. *The last proof shows that if $1 < q < p \leq 2$ and $b_p(x)$ has a last nonzero digit, then*

$$(3.1) \quad \lim_{y \nearrow x} b_{p,q}(y) > b_{p,q}(x).$$

Next we consider the same questions for the functions $a_{p,q}$. Now we have $p \in (1, 2]$ by definition.

Proposition 3.6. *Let $p \in (1, 2]$ and $q > 1$ with $p \neq q$.*

- (i) *If $q > p$, then $a_{p,q}$ is increasing.*
- (ii) *If $q < p$ then $a_{p,q}$ is nowhere monotone.*

Proof. (i) If $q > p$, then both a_p and the restriction of π_q to $R(a_p) \subseteq R(a_q)$ are increasing, hence $a_{p,q} = \pi_q \circ a_p$ is also increasing.

(ii) Let $q < p$, and fix an arbitrary open interval $I \subseteq [0, \frac{1}{p-1}]$ such that $I \cap J_p \neq \emptyset$.

Fix an arbitrary point $x \in I \cap J_p$, write $(a_i) = a_p(x)$ for brevity, and choose a large integer n such that $a_n = 1$, and $\pi_p(c) \in I$ for all p -quasi-greedy sequences c starting with $a_1 \cdots a_{n-1}0$.

Next write $(\alpha_i) = a_p(1)$ for brevity, and choose a large integer m such that

$$10^m \alpha_1 \alpha_2 \cdots < \alpha_1 \alpha_2 \cdots \quad \text{and} \quad \left(1 - \frac{1}{q^m}\right) \sum_{i=1}^{\infty} \frac{\alpha_i}{q^i} > 1.$$

This is possible because $\alpha_1 = 1$, $\alpha_1 \alpha_2 \cdots > 10^\infty$, and

$$\sum_{i=1}^{\infty} \frac{\alpha_i}{q^i} > \sum_{i=1}^{\infty} \frac{\alpha_i}{p^i} = 1.$$

It follows from the lexicographic characterization of quasi-greedy expansions and from our first assumption on m that

$$a_1 \cdots a_{n-1} 000 \alpha_1 \alpha_2 \cdots < a_1 \cdots a_{n-1} 00 \alpha_1 \alpha_2 \cdots < a_1 \cdots a_{n-1} 010^m \alpha_1 \alpha_2 \cdots$$

are quasi-greedy expansions in base p of suitable numbers $x < y < z$ (we use here the increasingness of the map $x \mapsto a_p(x)$). Since $x, y, z \in I$ by the choice of n , the proof will be completed by showing that $\pi_q(y) > \pi_q(x)$ and $\pi_q(y) > \pi_q(z)$.

The first relation is obvious:

$$\pi_q(y) - \pi_q(x) = \left(\frac{1}{q^{n+1}} - \frac{1}{q^{n+2}} \right) \sum_{i=1}^{\infty} \frac{\alpha_i}{q^i} > 0.$$

The second relation follows from our second assumption on m :

$$q^{n+1} (\pi_q(y) - \pi_q(z)) = \left(\left(1 - \frac{1}{q^m}\right) \sum_{i=1}^{\infty} \frac{\alpha_i}{q^i} \right) - 1 > 0. \quad \square$$

Corollary 3.7. *Let $p \in (1, 2]$ and $q > 1$ with $p \neq q$.*

- (i) $a_{p,q}$ is injective if and only if $q > p$.
- (ii) $a_{p,q}$ is onto if and only if $p > q$.
- (iii) $a_{p,q}$ is never bijective.

Proof. Part (i) readily follows from the preceding proposition.

Since π_q is a bijection between $R(a_q)$ and J_q , $a_{p,q} = \pi_q \circ a_p$ is onto if and only if $R(a_q) \subseteq R(a_p)$. In view of Proposition 2.4 this is equivalent to the condition $p > q$.

Parts (i) and (ii) imply (iii). \square

Proposition 3.8. *Let $p \in (1, 2]$ and $q > 1$ with $p \neq q$.*

The function $a_{p,q}$ is left continuous. It is right continuous in $x \in [0, \frac{1}{p-1})$ if and only if $b_p(x) = a_p(x)$.

Proof. Using the continuity of composite functions, Propositions 2.3 and 2.6 imply the positive continuity statements. It remains to prove that if $(b_i) := b_p(x)$ has a last non-zero digit $b_n = 1$, then $a_{p,q}$ is not right continuous in x .

Setting $(\alpha_i) := a_p(1)$ for brevity we have

$$b_p(x) = b_1 \cdots b_{n-1} b_n 0^\infty \quad \text{and} \quad a_p(x) = b_1 \cdots b_{n-1} (b_n - 1) \alpha_1 \alpha_2 \cdots .$$

Furthermore, if $y > x$ and $y \rightarrow x$, then $a_p(y) \rightarrow b_p(x)$ (see the proof of Proposition 2.6 (i)), and therefore

$$\begin{aligned} \lim_{y \searrow x} a_{p,q}(y) - a_{p,q}(x) &= \frac{1}{q^n} \left(1 - \sum_{i=1}^{\infty} \frac{\alpha_i}{q^i} \right) \\ &\neq \frac{1}{q^n} \left(1 - \sum_{i=1}^{\infty} \frac{\alpha_i}{p^i} \right) \\ &= 0. \end{aligned} \quad \square$$

Remark 3.9. *The last proof shows that if $1 < q < p \leq 2$ and $b_p(x)$ has a last nonzero digit, then*

$$(3.2) \quad \lim_{y \searrow x} a_{p,q}(y) < a_{p,q}(x).$$

4. DIFFERENTIABILITY AND BOUNDED VARIATION PROPERTY

For the proofs of this section we recall from [9, Lemmas 3.1 and 3.2] a property of greedy expansions:

Lemma 4.1. *Let $p \in (1, 2]$.*

- (i) *If $(c_i) = b_p(x)$ or $(c_i) = a_p(x)$ for some $x \in J_p$, then for each $n \geq 1$ there exists $x_n \in J_p$ such that $x_n \leq x$ and $b_p(x_n) = c_1 \cdots c_n 0^\infty$.*
- (ii) *If $b_p(x) \neq 1^\infty$ for some $x \in J_p$, then for each $n \geq 1$ there exists $x_n \in J_p$ such that $x_n > x$ and*

$$b_1(x_n, p) \cdots b_n(x_n, p) = b_1(x, p) \cdots b_n(x, p).$$

In the following proposition we use the Dini derivatives of a function, defined by the formulas

$$d_- f(x) := \liminf_{y \nearrow x} \frac{f(x) - f(y)}{x - y}, \quad D_- f(x) := \limsup_{y \nearrow x} \frac{f(x) - f(y)}{x - y}$$

and

$$d_+ f(x) := \liminf_{y \searrow x} \frac{f(x) - f(y)}{x - y}, \quad D_+ f(x) := \limsup_{y \searrow x} \frac{f(x) - f(y)}{x - y}.$$

We recall that f is differentiable in x if and only if all four Dini derivatives exist, are finite and are equal in a .

Proposition 4.2.

- (i) *If $1 < q < p \leq 2$, then $b_{p,q}$ and $a_{p,q}$ are not differentiable anywhere. More precisely, their Dini derivatives satisfy the following relations:*
 - (a) *If $x = 0$, then $d_+ b_{p,q}(x) = d_+ a_{p,q}(x) = \infty$.*
 - (b) *If $x = 1/(p-1)$, then $d_- b_{p,q}(x) = d_- a_{p,q}(x) = \infty$.*

(c) If $x \in (0, \frac{1}{p-1})$ and $b_p(x) = a_p(x)$, then $D_-b_{p,q}(x) = D_-a_{p,q}(x) = \infty$.

(d) If $b_p(x) \neq a_p(x)$, then

$$D_-b_{p,q}(x) = D_+a_{p,q}(x) = -\infty \quad \text{and} \quad D_+b_{p,q}(x) = D_-a_{p,q}(x) = \infty.$$

(ii) If $p \in (1, 2]$ and $q > p$, then $b_{p,q}$ and $a_{p,q}$ are differentiable almost everywhere.

(iii) If $p > 2$, then $B_{p,q}$ is differentiable with $B'_{p,q} > 0$ almost everywhere.

Proof. (i) Assume that $1 < q < p \leq 2$. First we investigate the left differentiability of $b_{p,q}$ in a point $x \in (0, \frac{1}{p-1}]$.

If $x = 1/(p-1)$, then $b_p(x) = 1^\infty$. If $x_n \nearrow x$, then $(b_{n,i}) := b_p(x_n)$ starts with $1^m 0$ where $m = m(n) \rightarrow \infty$ as $n \rightarrow \infty$. Therefore, writing $\bar{b}_i := 1 - b_i$,

$$\begin{aligned} \frac{b_{p,q}(x) - b_{p,q}(x_n)}{x - x_n} &= \frac{\pi_q(0^m 1 \bar{b}_{n,m+2} \bar{b}_{n,m+3} \cdots)}{\pi_p(0^m 1 \bar{b}_{n,m+2} \bar{b}_{n,m+3} \cdots)} \\ &\geq \frac{\pi_q(0^m 10^\infty)}{\pi_p(0^m 1^\infty)} \\ &= \frac{p-1}{p} \left(\frac{p}{q}\right)^{m+1} \end{aligned}$$

for each n . Since $p > q$, letting $n \rightarrow \infty$ this yields $d_-b_{p,q}(x) = \infty$.

If $x > 0$ and $b_p(x) = a_p(x)$, then $(b_i) := b_p(x)$ has infinitely non-zero digits $b_n = 1$. For each such n , applying Lemma 4.1 there exists $x_n < x$ such that $b_p(x_n) = b_1 \cdots b_{n-1} 0^\infty$. Then $x_n \rightarrow x$, and

$$\begin{aligned} \frac{b_{p,q}(x) - b_{p,q}(x_n)}{x - x_n} &= \frac{\pi_q(b_1 b_2 \cdots) - \pi_q(b_1 \cdots b_{n-1} 0^\infty)}{\pi_p(b_1 b_2 \cdots) - \pi_p(b_1 \cdots b_{n-1} 0^\infty)} \\ &= \frac{\pi_q(0^{n-1} 1 b_{n+1} \cdots)}{\pi_p(0^{n-1} 1 b_{n+1} \cdots)} \\ &\geq \frac{\pi_q(0^{n-1} 10^\infty)}{\pi_p(0^{n-1} 1^\infty)} \\ &= \frac{p-1}{p} \left(\frac{p}{q}\right)^n \end{aligned}$$

for each n . Letting $n \rightarrow \infty$ we conclude that $D_-b_{p,q}(x) = \infty$.

If $b_p(x) \neq a_p(x)$, then using (3.1) we obtain that

$$D_-b_{p,q}(x) = \lim_{y \nearrow x} \frac{b_{p,q}(x) - b_{p,q}(y)}{x - y} = -\infty.$$

Next we investigate the right differentiability of $b_{p,q}$ in a point $x \in [0, \frac{1}{p-1})$.

If $x = 0$, then $b_p(x) = 0^\infty$. If $x_n \searrow x$, then $(b_{n,i}) := b_p(x_n)$ starts with $0^m 1$ where $m = m(n) \rightarrow \infty$ as $n \rightarrow \infty$. Therefore

$$\frac{b_{p,q}(x_n) - b_{p,q}(x)}{x_n - x} = \frac{\pi_q(0^m 1 b_{n,m+2} b_{n,m+3} \cdots)}{\pi_p(0^m 1 b_{n,m+2} b_{n,m+3} \cdots)} \geq \left(\frac{p}{q}\right)^m \frac{\pi_q(10^\infty)}{\pi_p(1^\infty)}$$

for each n . Letting $n \rightarrow \infty$ we conclude that $d_+ b_{p,q}(0) = \infty$.

If $b_p(x) \neq a_p(x)$, then $(b_i) := b_p(x)$ has a last non-zero element $b_n = 1$. By Lemma 4.1 there exist arbitrarily large integers $m > n$ such that

$$b_1 \cdots b_n 0^{m-n} 10^\infty = b_p(x_m)$$

for some $x_m > x$. Then

$$\frac{b_{p,q}(x_m) - b_{p,q}(x)}{x_m - x} = \frac{\pi_q(0^m 10^\infty)}{\pi_p(0^m 10^\infty)} = \left(\frac{p}{q}\right)^m.$$

Letting $m \rightarrow \infty$ we conclude that $D_+ b_{p,q}(x) = \infty$.

Now we turn to the function $a_{p,q}$. First we investigate its left differentiability in a point $x \in (0, \frac{1}{p-1}]$.

If $x = 1/(p-1)$, then $a_p(x) = 1^\infty$. If $x_n \nearrow x$, then $(a_{n,i}) := a_p(x_n)$ starts with $1^m 0$ where $m = m(n) \rightarrow \infty$ as $n \rightarrow \infty$. Therefore, writing $\tilde{a}_i := 1 - a_i$,

$$\begin{aligned} \frac{a_{p,q}(x) - a_{p,q}(x_n)}{x - x_n} &= \frac{\pi_q(0^m 1 \overline{a_{n,m+2} a_{n,m+3} \cdots})}{\pi_p(0^m 1 \overline{a_{n,m+2} a_{n,m+3} \cdots})} \\ &\geq \frac{\pi_q(0^m 10^\infty)}{\pi_p(0^m 10^\infty)} \\ &= \frac{p-1}{p} \left(\frac{p}{q}\right)^{m+1} \end{aligned}$$

for each n . Since $p > q$, letting $n \rightarrow \infty$ this yields $d_- a_{p,q}(x) = \infty$.

The following consideration is valid for all $x \in (0, \frac{1}{p-1}]$. Set $(\alpha_i) := a_p(1)$. Furthermore, fix a number $r \in (1, q)$ and set $(\tilde{\alpha}_i) := a_r(1)$. Since $r < p$, we have $(\tilde{\alpha}_i) < (\alpha_i)$.

Since $x > 0$, $(a_i) := a_p(x)$ has infinitely many nonzero digits $a_n = 1$. For each such n , $a_1 \cdots a_n 0^\infty$ is p -greedy, hence $a_1 \cdots a_{n-1} 0 \alpha_1 \alpha_2 \cdots$ is p -quasi-greedy.

We claim that $(c_i) := a_1 \cdots a_{n-1} 0 \tilde{\alpha}_1 \tilde{\alpha}_2 \cdots$ is also p -quasi-greedy, and hence

$$a_p(x_n) = a_1 \cdots a_{n-1} 0 \tilde{\alpha}_1 \tilde{\alpha}_2 \cdots$$

for some $x_n < x$.

For this we have to show that

$$(c_{k+i}) \leq (\alpha_i) \quad \text{whenever} \quad c_k < 0.$$

We have

$$(c_{k+i}) = \begin{cases} a_{k+1} \cdots a_{n-1} 0 \tilde{\alpha}_1 \tilde{\alpha}_2 \cdots & \text{if } k < n \text{ and } a_k = 0, \\ \tilde{\alpha}_1 \tilde{\alpha}_2 \cdots & \text{if } k = n, \\ \tilde{\alpha}_{k-n+1} \tilde{\alpha}_{k-n+2} \cdots & \text{if } k > n \text{ and } \tilde{\alpha}_{k-n} = 0. \end{cases}$$

Since

$$\tilde{\alpha}_{j+1} \tilde{\alpha}_{j+2} \cdots \leq \tilde{\alpha}_1 \tilde{\alpha}_2 \cdots \leq \alpha_1 \alpha_2 \cdots$$

for $j = 0$ and for all $j \geq 1$ satisfying $\tilde{\alpha}_j = 0$,³ it follows that

$$(c_{k+i}) \leq \begin{cases} a_{k+1} \cdots a_{n-1} 0 \alpha_1 \alpha_2 \cdots & \text{if } k < n \text{ and } a_k = 0, \\ \alpha_1 \alpha_2 \cdots & \text{if } k \geq n \text{ and } c_k = 0. \end{cases}$$

We conclude by observing that

$$a_{k+1} \cdots a_{n-1} 0 \alpha_1 \alpha_2 \cdots \leq \alpha_1 \alpha_2 \cdots \quad \text{if } k < n \quad \text{and} \quad a_k = 0$$

because the sequence $a_1 \cdots a_{n-1} 0 \alpha_1 \alpha_2 \cdots$ is p -quasi-greedy.

It follows that

$$\begin{aligned} \frac{a_{p,q}(x) - a_{p,q}(x_n)}{x - x_n} &= \frac{\pi_q(a_1 a_2 \cdots) - \pi_q(a_1 \cdots a_{n-1} 0 \tilde{\alpha}_1 \tilde{\alpha}_2 \cdots)}{\pi_p(a_1 a_2 \cdots) - \pi_p(a_1 \cdots a_{n-1} 0 \tilde{\alpha}_1 \tilde{\alpha}_2 \cdots)} \\ &= \left(\frac{p}{q}\right)^{n-1} \frac{\pi_q(1 a_{n+1} a_{n+2} \cdots) - \pi_q(0 \tilde{\alpha}_1 \tilde{\alpha}_2 \cdots)}{\pi_p(1 a_{n+1} a_{n+2} \cdots) - \pi_p(0 \tilde{\alpha}_1 \tilde{\alpha}_2 \cdots)} \\ &\geq \left(\frac{p}{q}\right)^{n-1} \frac{\pi_q(10^\infty) - q^{-1} \pi_q(\tilde{\alpha}_1 \tilde{\alpha}_2 \cdots)}{\pi_p(1^\infty)} \\ &= \left(\frac{p}{q}\right)^{n-1} \frac{p-1}{q} (1 - \pi_q(\tilde{\alpha}_1 \tilde{\alpha}_2 \cdots)) \end{aligned}$$

for each n . The right-hand side tends to infinity as $n \rightarrow \infty$ because $r < q$, and hence

$$\pi_q(\tilde{\alpha}_1 \tilde{\alpha}_2 \cdots) < \pi_r(\tilde{\alpha}_1 \tilde{\alpha}_2 \cdots) = 1.$$

Therefore $D_- a_{p,q}(x) = \infty$.

Finally we investigate the right differentiability of $a_{p,q}$ in a point $x \in [0, \frac{1}{p-1}]$.

If $x = 0$, then $a_p(x) = 0^\infty$. If $x_n \searrow x$, then $(a_{n,i}) := a_p(x_n)$ starts with $0^m 1$ where $m = m(n) \rightarrow \infty$ as $n \rightarrow \infty$. Therefore

$$\frac{a_{p,q}(x_n) - a_{p,q}(x)}{x_n - x} = \frac{\pi_q(0^m 1 a_{n,m+2} a_{n,m+3} \cdots)}{\pi_p(0^m 1 a_{n,m+2} a_{n,m+3} \cdots)} \geq \left(\frac{p}{q}\right)^m \frac{\pi_q(10^\infty)}{\pi_p(1^\infty)}$$

for each n . Letting $n \rightarrow \infty$ we conclude that $d_+ a_{p,q}(0) = \infty$.

If $b_p(x)$ has a last non-zero digit $b_n = 1$, then using relation (3.2) we obtain that

$$D_+ a_{p,q}(x) = \lim_{y \searrow x} \frac{a_{p,q}(x) - a_{p,q}(y)}{x - y} = -\infty.$$

(ii) If $p \in (1, 2]$ and $q > p$, then $b_{p,q}$ is increasing Proposition 3.1 (i), and we may apply the Lebesgue differentiability theorem.

(iii) If $p > 2$, then J_p is a null set, and $B_{p,q}$ is differentiable in each point $x \in [0, \frac{1}{p-1}] \setminus J_p$, because it is linear by definition in a neighborhood of x . \square

In the following proposition we write $B_{p,q}$ instead of $b_{p,q}$ when $p \in (1, 2]$, so that $B_{p,q} : [0, \frac{1}{p-1}] \rightarrow [0, \frac{1}{q-1}]$ for all $p, q > 1$.

³In fact for all $j \geq 0$ by the results of [24] and [12].

Proposition 4.3.

- (i) $B_{p,q}$ is continuous if and only if $p > 2$.
- (ii) $B_{p,q}$ has bounded variation if and only if $q \geq \min\{p, 2\}$.
- (iii) $B_{p,q}$ is absolutely continuous if and only if $\min\{p, q\} > 2$.
- (iv) If $p > 2$, then $B_{p,q}$ is Hölder continuous with the exact exponent $\min\left\{1, \frac{\log q}{\log p}\right\}$.

Remark 4.4. Comparing Propositions 4.2 and 4.3 we see that in case $1 < q < 2 < p$ the function $B_{p,q}$ is differentiable almost everywhere, although it has no bounded variation. This is due to the artificial linear extension over J_p .

Proof of Proposition 4.3. (i) This follows from Proposition 3.3.

(ii) If $q \geq \min\{p, 2\}$, then $B_{p,q}$ is even non-decreasing by Proposition 3.1 and by the affine nature of the extension defining $B_{p,q}$.

Assume henceforth that $q < \min\{p, 2\}$. Fix a number r such that $q < r < \min\{p, 2\}$ and $(\beta_i) := b_r(1)$ has a last non-zero digit $\beta_m = 1$. This is possible by [24, Lemma 3.1].

Fix a positive integer n and consider an arbitrary r -greedy sequence $b_1 \cdots b_n \in \{0, 1\}^n$. Then

$$b_1 \cdots b_n 0^\infty < b_1 \cdots b_n 0^m 10^\infty$$

are r -greedy and hence also p -greedy sequences, so that

$$\pi_p(b_1 \cdots b_n 0^\infty) < \pi_p(b_1 \cdots b_n 0^m 10^\infty).$$

On this interval the total variation of $B_{p,q}$ is at least

$$\pi_q(b_1 \cdots b_n 0^m 10^\infty) - \pi_q(b_1 \cdots b_n 0^\infty) = \pi_q(0^{n+m} 10^\infty) = \frac{1}{q^{n+m+1}}.$$

Now we recall from [31, p. 490] that there are at least r^n r -greedy sequences $b_1 \cdots b_n \in \{0, 1\}^n$ for each n . Hence the total variation of $B_{p,q}$ is at least

$$\sup_n \frac{r^n}{q^{n+m+1}} = \frac{1}{q^{m+1}} \left(\frac{r}{q}\right)^n = \infty.$$

(iii) In view of (i) and (ii) it suffices to investigate the absolute continuity for $p > 2$ and $q \geq 2$. Since in this case $B_{p,q}$ is non-decreasing by Proposition 3.1, it is absolute continuous if and only if the Newton–Leibniz formula holds:

$$\int_0^{1/(p-1)} B'_{p,q}(x) dx = B_{p,q}(1/(p-1)) - B_{p,q}(0).$$

The following computation is valid for all $p > 2$ and $q > 1$, and it also illustrates the non-monotonicity for $1 < q < 2$.

We are going to show that

$$\int_0^{1/(p-1)} B'_{p,q}(x) dx = \begin{cases} 1/(q-1) & \text{if } q > 2, \\ 0 & \text{if } q = 2, \\ -\infty & \text{if } 1 < q < 2. \end{cases}$$

Since

$$B_{p,q}(1/(p-1)) - B_{p,q}(0) = \frac{1}{q-1},$$

the Newton–Leibniz formula holds if and only if $q > 2$.

Henceforth we write $f := B_{p,q}$ for brevity. We consider the connected components of $[0, \frac{1}{p-1}] \setminus J_p$ in the Cantor type construction of J_p , and we denote by $I_{m,k}$ the closures of these intervals for $m = 0, 1, \dots$ and $k = 1, \dots, 2^m$. For example,

$$I_{0,1} := \left[\frac{1}{p(p-1)}, \frac{1}{p} \right]$$

for $m = 0$ and

$$I_{1,1} := \left[\frac{1}{p^2(p-1)}, \frac{1}{p^2} \right], \quad I_{1,2} := \left[\frac{1}{p} + \frac{1}{p^2(p-1)}, \frac{1}{p} + \frac{1}{p^2} \right]$$

for $m = 1$. It follows from the construction that the length of $I_{m,k}$ is equal to

$$(4.1) \quad |I_{m,k}| = \frac{p-2}{p^{m+1}(p-1)}$$

for all m, k .

Since $B_{p,q}$ is affine on each interval $I_{m,k}$, the integral of $B_{p,q}$ over $I_{m,k}$ may be computed by using the Newton–Leibniz formula. For example,

$$(4.2) \quad \begin{aligned} \int_{I_{m,1}} B'_{p,q}(x) dx &= \int_{\frac{1}{p^{m+1}(p-1)}}^{\frac{1}{p^{m+1}}} B'_{p,q}(x) dx \\ &= B_{p,q}\left(\frac{1}{p^{m+1}}\right) - B_{p,q}\left(\frac{1}{p^{m+1}(p-1)}\right) \\ &= \frac{1}{q^{m+1}} - \frac{1}{q^{m+1}(q-1)} \\ &= \frac{q-2}{q^{m+1}(q-1)} \end{aligned}$$

for all m . By translation invariance the integrals over $I_{m,k}$ do not depend on k . Therefore, taking into account that J_p is a null set, we obtain that

$$\int_0^{1/(p-1)} B'_{p,q}(x) dx = \sum_{m=0}^{\infty} \sum_{k=1}^{2^m} B'_{p,q}(x) dx = \sum_{m=0}^{\infty} 2^m \frac{q-2}{q^{m+1}(q-1)}.$$

For $q = 2$ each term, and hence the integral vanishes. For $1 < q < 2$ the general term tends to $-\infty$, so that the integral is equal to $-\infty$. For $q > 2$ we have a convergent geometric series, and

$$\int_0^{1/(p-1)} B'_{p,q}(x) dx = \frac{q-2}{q(q-1)} \sum_{m=0}^{\infty} \left(\frac{2}{q}\right)^m = \frac{1}{1-\frac{2}{q}} \cdot \frac{q-2}{q(q-1)} = \frac{1}{q-1}.$$

(iv) The Hölder exponent of the extended function $B_{p,q}$ cannot be larger than the Hölder exponent $\alpha := \frac{\log q}{\log p}$ of $b_{p,q}$, obtained in Proposition 3.3, and it cannot be larger than 1 because $B_{p,q}$ is defined on an interval. It remains to show that $B_{p,q}$ is Hölder continuous with the exponent $\min\{1, \alpha\}$.

In case $q > p > 2$ we have to prove that $B_{p,q}$ is Lipschitz continuous. Since it is affine on each intervals $I_{m,k}$, it follows from the estimates (4.1), (4.2) and from the above mentioned translation invariance that

$$(4.3) \quad B'_{p,q}(x) = \frac{q-2}{q^{m+1}(q-1)} \cdot \frac{p^{m+1}(p-1)}{p-2}$$

on each $I_{m,k}$. Since $q > p > 2$, all these derivatives are positive and uniformly bounded:

$$0 < \frac{q-2}{q^{m+1}(q-1)} \cdot \frac{p^{m+1}(p-1)}{p-2} \leq \frac{p(p-1)(q-2)}{q(q-1)(p-2)}.$$

Since J_p is a null set, this shows that the absolutely continuous function $B_{p,q}$ has an a.e. bounded derivative.

Next we assume that $p > \max\{q, 2\}$. We already know that $B_{p,q}$ is Hölder continuous on J_p with the exponent $\alpha < 1$ and some constant c_1 .

We claim that $B_{p,q}$ is also Hölder continuous on each interval $I_{m,k}$ with the same exponent α and with some constant c_2 independent of m and k .

Since

$$|B_{p,q}(x) - B_{p,q}(y)| \leq \frac{p^{m+1}(p-1)(q-2)}{q^{m+1}(q-1)(p-2)} |x - y|$$

for all $x, y \in I_{m,k}$ by (4.3), it suffices to find a constant c_2 satisfying

$$\frac{p^{m+1}(p-1)(q-2)}{q^{m+1}(q-1)(p-2)} |x - y| \leq c_2 |x - y|^\alpha$$

for all $x, y \in I_{m,k}$, or equivalently that

$$\frac{p^{m+1}(p-1)(q-2)}{q^{m+1}(q-1)(p-2)} |I_{m,k}|^{1-\alpha} \leq c_2$$

for all m, k . Since $p^\alpha = q$ by the definition of α , this is satisfied:

$$\begin{aligned} & \frac{p^{m+1}(p-1)(q-2)}{q^{m+1}(q-1)(p-2)} |I_{m,k}|^{1-\alpha} \\ &= \left(\frac{p}{q}\right)^{m+1} \cdot \frac{p-1}{q-1} \cdot \frac{q-2}{p-2} \left(\frac{p-2}{p^{m+1}(p-1)}\right)^{1-\alpha} \\ &= \frac{q-2}{q-1} \left(\frac{p-1}{p-2}\right)^\alpha =: c_2. \end{aligned}$$

The estimate

$$|f(x) - f(y)| \leq c_2 |x - y|^\alpha$$

remains valid by continuity on the closure of each interval $I_{m,k}$.

Finally we prove that

$$|f(x) - f(y)| \leq (c_1 + 2c_2) |x - y|^\alpha$$

for all $x, y \in [0, \frac{1}{p-1}]$.

Since $[0, \frac{1}{p-1}] \setminus J_p$ is dense in $[0, \frac{1}{p-1}]$ (because J_p is a null set), we may assume by continuity that $x, y \in [0, \frac{1}{p-1}] \setminus J_p$, and we may assume by symmetry that $x < y$.

If x and y belong to the same connected component $I_{m,k}$, then we already know this inequality with c_2 in place of $c_1 + 2c_2$. Otherwise we have $x \in I_{m,k}$ and $y \in I_{n,\ell}$ with different connected components. In this case we introduce the right endpoint u of $I_{m,k}$, the left endpoint v of $I_{n,\ell}$, and we conclude as follows:

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(u)| + |f(u) - f(v)| + |f(v) - f(y)| \\ &\leq c_2 |x - u|^\alpha + c_1 |u - v|^\alpha + c_2 |v - y|^\alpha \\ &\leq (c_2 + c_1 + c_2) |x - y|^\alpha. \end{aligned} \quad \square$$

If $p > q = 2$, then $B_{p,q}$ is non-decreasing, continuous, but not absolutely continuous by Propositions 3.1 and 4.3.⁴ Its arc length cannot be larger than $1 + \frac{1}{p-1}$.

Indeed, if, more generally, $f : [a, b] \rightarrow [c, d]$ is a non-decreasing function, then, following [19] for any finite subdivision $a = x_0 < \dots < x_n = b$ we have

$$\begin{aligned} & \sum_{i=1}^n \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\ & \leq \sum_{i=1}^n ((x_i - x_{i-1}) + (f(x_i) - f(x_{i-1}))) \\ & = (b - a) + (f(b) - f(a)) \\ & \leq (b - a) + (d - c). \end{aligned}$$

In fact, for $B_{p,2}$ this maximum is achieved:

⁴We recall that Cantor's ternary function corresponds to the case $p = 3$.

Proposition 4.5. *If $p > 2$, then $B_{p,2}$ is continuous, non-decreasing and its arc length is equal to $p/(p-1)$.*

Proof. We already know from Proposition 4.3 that $B_{p,2}$ is continuous and non-decreasing. We approximate the arc length of $B_{p,2}$ by a sequence of polygonal lines as follows. We use the intervals $I_{m,k}$ from the preceding proof.

The length of the polygonal line determined by the endpoints of the intervals $[0, 1/(p-1)]$ and $I_{0,1} = [1/(p(p-1)), 1/p]$ is equal to

$$L_1 := \frac{p-2}{p(p-1)} + 2\sqrt{\left(\frac{1}{p(p-1)}\right)^2 + \frac{1}{4}}.$$

If we add the four endpoints of the intervals $I_{1,1}$ and $I_{1,2}$, then we obtain the arclength

$$L_2 = \frac{p-2}{p(p-1)} + \frac{2(p-2)}{p^2(p-1)} + 4\sqrt{\left(\frac{1}{p^2(p-1)}\right)^2 + \frac{1}{4^2}}.$$

At the next step we add the endpoints of four intervals $I_{2,1}, \dots, I_{2,4}$ to obtain the arclength

$$L_3 = \frac{p-2}{p(p-1)} + \frac{2(p-2)}{p^2(p-1)} + \frac{4(p-2)}{p^3(p-1)} + 8\sqrt{\left(\frac{1}{p^3(p-1)}\right)^2 + \frac{1}{4^3}}.$$

Continuing by induction we obtain

$$L_n = 2^n \sqrt{\left(\frac{1}{p^n(p-1)}\right)^2 + \frac{1}{4^n}} + \sum_{k=1}^n \frac{2^{k-1}(p-2)}{p^k(p-1)}$$

for $n = 1, 2, \dots$.

Letting $n \rightarrow \infty$ we obtain that the arc length of $B_{p,2}$ is at least

$$\lim L_n = 1 + \frac{p-2}{p(p-1)} \sum_{k=1}^{\infty} (2/p)^{k-1} = 1 + \frac{p-2}{p(p-1)} \cdot \frac{1}{1 - (2/p)} = \frac{p}{p-1}.$$

In view of the preceding remark we have equality here. \square

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